# properties of Laplace Transform.

1) If 'a' is a constant and F(s) is the laplace transform of f(t) then,

$$L\{a f(t)\} = a F(s)$$

2) If $f_1(t)$ and $f_2(t)$ are two time functions and $F_1(s)$ and $F_2(s)$ are their respective Laplace transform then,

$$L\{f_1(t)\} \pm L\{f_2(t)\} = F_1(s) \pm F_2(s)$$

3) Time shift property: If 'a' is a positive real number and F(s) is the Laplace transform of f(t) then,

$$L\{f(t-a)\} = e^{-as} F(s)$$

Thus, if the original time function is shifted by to the right by 'a' on time axis, its Laplace transform is equal to the original Laplace transform multiplied by $e^{-as}$.

~~for example~~:

4) If $F(s)$ is the Laplace transform of $f(t)$ then

$$L\{e^{at}f(t)\} = F(s-a)$$

similarly,

$$L\{e^{-at}f(t)\} = F(s+a)$$

e.g. find the laplace transform of $e^{-at}\sin\omega t$

∴ we know that

$$L\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

then

$$L\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2} \quad \underline{Ans}$$

5) Real differentiation theorem: The Laplace transform of the derivatives of function f(t) is given by

$$L\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0)$$

where

$f(0)$ is the initial value of function at $t=0$

6) Laplace transform of higher derivatives:

$$L\left\{\frac{d^n}{dt^n}f(t)\right\} = s^nF(s) - s^{n-1}f(0) \cdots\cdots \frac{d^{n-1}f(0)}{dt^{n-1}}$$

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where $f(0)$ is the value of $f(t)$ at $t=0$ also $\frac{d^{n-1} f(0)}{dt^{n-1}}$ is the value of $\frac{d^{n-1} f(t)}{dt^{n-1}}$ at $t=0$

7) Laplace transform of integral of $f(t)$

$$L\left\{ \int f(t)\, dt \right\} = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

where

$f^{-1}(0) = \int f(t)\, dt$ at $t=0$.

8) Complex differentiation theorem:

According to this theorem

$$L\{ t^n f(t) \} = (-1)^n \frac{d^n}{ds^n} F(s)$$

e.g. Find the Laplace Transform of $t \sinh at$

Sol$^n$: Using the property Laplace transform we get.

$$\therefore L\{ t^n f(t) \} = (-1)^n \frac{d^n F(s)}{dt^n} \qquad \because L[f(t)] = \frac{a}{s^2-a^2}$$

$$= (-1)^1 \frac{d\left(\frac{a}{s^2-a^2}\right)}{ds}$$

$$= (-1)^1 \frac{a}{} \left[ \frac{2s}{(s^2-a^2)^2} \right]$$

$$= -1 \frac{d\left[\frac{a}{(s^2-a^2)}\right]}{ds}$$

$$= - \left[ \frac{0(s^2-a^2) - a \cdot 2s}{(s^2-a^2)^2} \right]$$

$$= \frac{2as}{(s^2-a^2)^2} \text{ Ans}$$

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9) Complex integration:

If $F(s)$ is the laplace transform of $f(t)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

e.g., calculate the L.T of $\frac{\sin 2t}{t}$

∴ we know that $L[\sin 2t] = \frac{2}{s^2+4}$

$$\therefore L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{(s^2+4)}\,ds$$

$$= 2\int_s^\infty \frac{1}{(s^2+4)}\,ds$$

$$= 2\left[\frac{1}{2}\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \tan^{-1}\frac{\infty}{2} - \tan^{-1}\frac{s}{2}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2} \quad \text{Ans}$$

10) Initial value theorem:

If initial value of the function $f(t)$ is required, then it can be calculated by the initial value theorem which states that

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)}$$

For example. find the initial value of $e^{-at}$

Soln:- $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

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$\left[\frac{1}{\infty} = 0\right]$

$$\therefore L\{e^{-at}\} = \int_0^\infty e^{-(s+a)t}\,dt = -\frac{\left[e^{-(s+a)t}\right]_0^\infty}{(s+a)}$$

$$= -\frac{[0-1]}{(s+a)}$$

$$= \frac{1}{(s+a)}$$

$$\therefore \lim_{s\to\infty} s\cdot F(s) =$$

$$\lim_{s\to\infty} s\cdot\frac{1}{(s+a)} = \lim_{s\to\infty}\frac{1}{1+\frac{a}{s}}$$

$$= \frac{1}{1+0} = 1 \text{ Ans}$$

11) Final value theorem: If the final value of the function $f(t)$ is required then it can be calculated by using final value theorem, which states that—

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)}$$

e.g. find the final value of $e^{-at}$

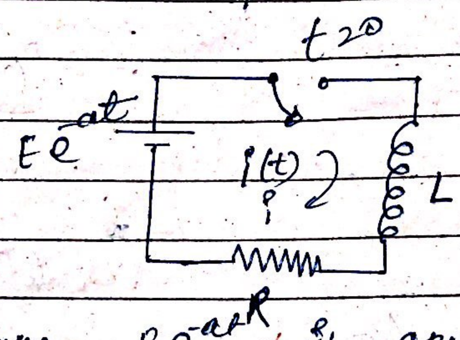
Soln: $\therefore L\{e^{-at}\} = \frac{1}{(s+a)}$

$$\therefore \lim_{s\to 0} s\cdot\frac{1}{(s+a)} = \lim_{s\to 0}\frac{1}{1+\frac{a}{s}}$$

$$= \frac{1}{1+\infty} = \frac{1}{\infty} = 0 \text{ Ans}$$

## Application of Laplace Transform in solving the electrical circuit.

**Q** Consider the R-L circuit as shown below.




If a voltage $Ee^{-at}$ is applied at $t = 0$ to the circuit containing L and R show that current at time 't' is $\frac{E}{R-aL}\left(e^{-at} - e^{-\frac{Rt}{L}}\right)$.

Solution: Using the Kirchhoff's law we get

$$E(t) = Ri + L\frac{di}{dt} \quad \text{---} (I)$$

$$Ee^{-at} = Ri + L\frac{di}{dt} \quad \text{---} (II)$$

Here $i(0) = 0$ Initial value.

Now applying Laplace transform on the both sides we get:

$$L\left[Ee^{-at}\right] = \left[Ri(t)\right] + L\left[L\frac{di(t)}{dt}\right]$$

$$E\frac{1}{(S+a)} = RI(s) + LsI(s) - LsI(0)$$

$$\frac{E}{(S+a)} = RI(s) + LsI(s)$$

$$= I(s)(R+Ls)$$

$$I(s) = \frac{E}{(S+a)(R+LS)}$$

$$I(s) = \frac{E}{(S+a)(LS+R)}$$

Now, taking inverse square law we get.

$$I(s) = E\left[\frac{A}{(S+a)} + \frac{B}{(LS+R)}\right]$$

~~$E = (LS+R)A + B(S+a)$~~

using partial fraction we get

~~$A = \frac{E}{(R-aL)}$, $B = \frac{EL}{(R-aL)}$~~

~~$\therefore\ \mathcal{L}(i) = E\left[\frac{1}{(R-aL)} + \frac{L}{}\right]$~~

~~put $A = \frac{-R}{L}$~~

~~$E = 0 + B\left(\frac{-R}{L} + a\right)$~~

~~$E = B\frac{(aL-R)}{L}$~~

~~$B = \frac{EL}{(aL-R)}$~~

~~put $s = -a$~~

~~$E$~~

$$\frac{E}{(S+a)(LS+R)} = \frac{A}{(S+a)} + \frac{B}{(LS+R)} = \frac{A(LS+R)+B(S+a)}{(S+a)(LS+R)}$$

Now, put $S = -a$

$$E = A(LS+R) + B(S+a) \quad \text{---(1)}$$

put $S = -a$

$$E = A(-aL+R) + 0$$

$$A = \frac{E}{(R-aL)}$$

Now put $S = \frac{-R}{L}$

then, $E = 0 + B\left(\frac{-R}{L} + a\right) = B\frac{(aL-R)}{L}$

$$B = \frac{E}{(aL-R)}$$

$$\therefore\ \frac{E}{(S+a)(LS+R)} = \frac{E}{(R-aL)(S+a)} + \frac{EL}{(aL-R)}\cdot\frac{1}{(LS+R)}$$

$$= \frac{E}{\cancel{(aL-R)}}$$

$$I(s) = \frac{E}{(R-aL)}\left[\frac{1}{(s+a)} - \frac{L}{Ls+R}\right]$$

Now, taking the inverse laplace transform we get

$$i(t) = \frac{E}{(R-aL)}\left[e^{-at} - L\cdot e^{-\frac{R}{L}t}\right]$$

$$i(t) = \frac{E}{(R-aL)}\left(e^{-at} - L e^{-\frac{R}{L}t}\right) \text{ proved}$$